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Expansion of a Polynomial Raised to a Positive Integer Power

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Abstract

Background: The expansion of polynomial expressions raised to a positive integer power is a fundamental topic in algebra and combinatorics. Newton's Binomial Theorem provides a direct method for expanding expressions containing two variables, $(x_1 + x_2)^n$. However, no simple general formula is available for expressions involving more than two variables, such as $(x_1 + x_2 + \cdots + x_m)^n$. Consequently, expanding multinomial expressions becomes increasingly complex as the number of variables increases. Therefore, there is a need to develop a generalized mathematical formula that can determine the expansion, coefficients, and total number of terms for any positive integer value of m. This research aims to derive such a general formula by analyzing lower-dimensional cases and extending the observed patterns to the general multinomial case.

Objective: To be able to find the expansion in a concise and simple way.

Importance: Use of the expansion in some applications of differential and integral calculus and other topics.

Methodology: The study employed an analytical mathematical approach based on pattern recognition and mathematical induction. The procedure consisted of four main steps:

1. The classical binomial theorem ($m = 2$) was reviewed to identify the structure of the expansion, coefficients, and total number of terms.
2. The trinomial case ($m = 3$) was analyzed by classifying terms according to the number of variables involved and determining the corresponding coefficients using combinatorial principles.
3. The same analytical procedure was extended to the four-variable case ($m = 4$) to verify the emerging mathematical pattern.

4. Based on these observations, a generalized multinomial expansion formula was derived for any positive integer m, together with expressions for the multinomial coefficients and the total number of terms.

Results: The analysis demonstrated that the binomial expansion can be successfully generalized to polynomials containing any number of variables. Explicit expansions were obtained for the cases $m = 2$, $m = 3$, and $m = 4$, revealing a consistent mathematical structure. The coefficients were shown to follow multinomial coefficient rules, while the exponents satisfy the condition that their sum equals n. Furthermore, a general expression was derived for the total number of distinct terms in the expansion

$$\boxed{\frac{(n+m-1)!}{n!\,(m-1)!}}$$

The final generalized expansion was obtained as

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{p_1+p_2+\cdots+p_m=n} \frac{n!}{p_1!\,p_2!\cdots p_m!} x_1^{p_1} x_2^{p_2} \cdots x_m^{p_m},$$

where $p_1, p_2, \ldots, p_m \geq 0$.

Conclusion: This study successfully generalized the classical Binomial Theorem to the expansion of polynomials containing any number of variables. By examining the cases $m = 2, m = 3$, and $m = 4$, a consistent mathematical pattern was identified and extended to derive a general multinomial expansion formula. The resulting expression provides both the multinomial coefficients and the total number of terms for any positive integers $m$and n. The proposed formulation offers a simple and systematic method for expanding multinomial expressions and can serve as a useful tool in combinatorics, algebra, probability theory, and related areas of applied mathematics.

MSC 2020: Primary 05A10; Secondary 05A19, 11B65.

Keywords: Polynomial expansion; Positive integer power; Multinomial theorem; Multinomial coefficients; Algebraic identities; Polynomial identities.

Introduction:

The expansion of a polynomial raised to a positive integer power is one of the fundamental topics in algebra and combinatorial mathematics. It plays a crucial role in simplifying algebraic expressions, deriving mathematical identities, and solving a wide range of problems in pure and applied mathematics. Polynomial expansions are closely connected with the binomial theorem, multinomial theorem, combinatorial coefficients, and generating functions, making them an essential tool in mathematical analysis and computational methods [1]. Traditionally, the expansion of a polynomial raised to an integer power is obtained by repeated multiplication or by applying the multinomial theorem [2]. Although these classical methods provide accurate results, they become increasingly complicated and computationally expensive as the degree of the polynomial or the exponent increases [3]. The rapid growth in the number of terms and multinomial coefficients often makes the expansion process time-consuming and difficult to manage manually [4]. For this reason, many mathematicians have focused on developing alternative methods that simplify the expansion process while reducing the computational effort [5]. Several approaches have been proposed to determine the coefficients directly without performing successive multiplications [6]. These methods are particularly valuable in symbolic computation, computer algebra systems, numerical analysis, and mathematical software, where efficiency and accuracy are of great importance [7]. The study of polynomial expansions has numerous applications in different branches of mathematics and science, including algebra, combinatorics, numerical analysis, probability theory, mathematical modeling, engineering, computer science, and cryptography. Efficient expansion formulas contribute to improving algorithmic performance and facilitate the solution of complex mathematical problems [8]. The present study aims to develop a simple and efficient approach for expanding a polynomial raised to a positive integer power. The proposed method seeks to establish a general mathematical formula that enables the coefficients of the expanded polynomial to be determined directly, thereby minimizing computational complexity and avoiding repeated multiplication. It is expected that the proposed approach will provide an effective mathematical tool for researchers and students working in algebra and related fields.

Methodology

1. This study employed a mathematical and analytical approach to derive a generalized formula for the expansion of the polynomial $(x_1 + x_2 + \cdots + x_m)^n$, where $n$is a positive integer and $m \geq 2$. The methodology consisted of sequentially analyzing polynomial expansions with an increasing number of variables to identify recurring algebraic patterns.
2. Initially, the classical case of two variables ($m = 2$) was examined using Newton's Binomial Theorem as the fundamental basis. The structure of the expansion, the binomial coefficients, and the total number of terms were analyzed to establish the mathematical framework.
3. Subsequently, the trinomial case ($m = 3$) was investigated by expanding $(x_1 + x_2 + x_3)^n$. The resulting terms were classified according to the number of variables appearing in each term, and the relationships among coefficients, exponents, and the total number of terms were identified. Combinatorial principles were then used to express these coefficients in a systematic form.
4. The same analytical procedure was extended to the four-variable case ($m = 4$) to verify the consistency of the observed patterns. The expansion confirmed that the coefficient structure follows a predictable combinatorial arrangement, allowing the formulation of a generalized mathematical expression.
5. Finally, the patterns obtained from the cases $m = 2$, $m = 3$, and $m = 4$ were generalized to derive the multinomial expansion for any number of variables $m = 4$ The coefficient of each term was expressed using factorial notation,
6. $\frac{n!}{p_1! p_2! \cdots p_m!}$,
7. subject to the condition
8. $p_1 + p_2 + \cdots + p_m = n$,
9. thereby providing a unified mathematical formula for determining all terms and coefficients in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$. This methodology enabled the development of a general solution without requiring separate derivations for each specific value of m.

Result:

$(x_1 + x_2 + \ldots\ldots\ldots + x_{m)\ n}$ $\qquad n \in z^+$

-What we have available is only when m=2 which are called: binomial theory) Newton's Theorem)

- So is it possible to find a mathematical formula through which we can calculate the result

$$(X_1+x_2 + \ldots\ldots\ldots + x_m)^n$$

The answer is yes, and this research provides a mathematical formula for any value of m, where m= 2, 3, …

Research Method:

1- The formula is available when m=2
2-We study the case when m=3
3-We study the case when m=4
4-Through this we generalize the formula for any value of m

Research: **First**
We have when m=2

$$(x_1 + x_2)^n= x_1^n + c_1^n x_1^{n-1} x_2 + c_2^n x_1^{n-2} x_2^2 +\ldots\ldots\ldots\ldots\ldots\ldots+ c_{n-1}^n x_1 x_2^{n-1} + x_2^n$$

1- This result includes two terms with one symbol, namely x_1^n, x_{n2}

2- In the result there are (n-1) terms with the symbols x_1x_2

3-The total number of terms

$$n+1=c_1^{n+1}= c_{2-1}^{n+(2-1)}= c_{m-1}^{n+(m-1)}$$

Second:

We study the case when m=3

$$(x_1 + x_2 + x_3)^n = [x_1 + (x_2 + x_3)]^n$$

$$= x_1^n + c^n_1 x_1^{n-1}(x_2 + x_3) + c^n_2 x_1^{n-2}(x_2 + x_3)^2 + c^n_3 x_1^{n-3}(x_2 + x_3)^3 + \dots\dots\dots + c^n_{n-1} x_1 (x_2 + x_3)^{n-1} + (x_2 + x_3)^n$$

We assume I_K represents the order of the terms in the result, and V_K represents the number of terms in each one of them, so it becomes:

$I_1 = x_1^n$ $\qquad V=1$

$I_2 = c^n_1 x_1^{n-1}(x_2 + x_3) = c^n_1 x_1^{n-1} x_2 + c^n_1 x_1^{n-1} x_3$ $\qquad V_2 = 2$

$I_3 = c^n_2 x_1^{n-2}(x_2 + x_3)^2 = c^n_2 x_1^{n-2}(x_2^2 + c^2_1 x_2 x_3 + x_3^2) = c^n_2 x_1^{n-2} x_2^2 + c^2_1 c^n_2 x_1^{n-2} x_2 x_3 + c^n_2 x_1^{n-2} x_3^2$ $\qquad V_3=3$

$I_4 = C^n_3 x_1^{n-3}(x_2 + x_3)^3 = c^n_3 x_1^{n-3}(x_2^3 + c^3_1 x_2^2 x_3 + c^3_2 x_2 x_3^2 + x_3^2) = c^n_3 x_1^{n-3} x_2^3 + c^3_1 c^n_3 x_1^{n-3} x_2^2 x_3 + c^3_2 c^n_3 x_1^{n-3} x_2 x_3^2$
$c^n_3 x_1^{n-3} x_3^3$ $\qquad V_4=4$

....

$I_n = c^n_{n-1} x_1 (x_2 + x_3)^{n-1} = c^n_{n-1} x_1 (x_2^{n-1} + c^{n-1}_1 x_2^{n-2} x_3 + c^{n-1}_2 x_2^{n-3} x_3^2 + \dots\dots\dots c^{n-1}_{n-2} x_2 x_3^{n-2} + x_3^{n-1})$

$= c^n_{n-1} x_1 x_2^{n-1} + c^{n-1}_1 c^n_{n-1} x_1 x_2^{n-2} x_3 + c^{n-1}_2 c^n_{n-1} x_1 x_2^{n-3} x_3^2 + \dots\dots + c^{n-1}_{n-2} c^n_{n-1} x_1 x_2 x_3^{n-2} + c^n_{n-1} x_1 x_3^{n-1}$ $\qquad v_n = n$

$I_{n+1} = (x_2 + x_3)^n = x_2^n + c^n_1 x_2^{n-1} x_3 + \dots\dots\dots + c^n_{n-1} x_2 x_3^{n-1} + x_3^n$

$v_{n+1} = n+1$

The number of terms in the result $v_1 + v_2 + \dots\dots V_{n+1}$

$=1+2+3+\dots\dots\dots + n+1$

This represents an arithmetic sequence whose first term = 1 last term = n+1 number of terms = n+1 Thus the sum of the terms

$$S_{n+1} = \frac{n+1}{2}(1+n+1) = \frac{(n+1)(n+2)}{2}$$

In I_n we find

$$C^{n-1}_{\boxed{1}}\, C^{n}_{n-1}\, x_1\, x_2^{n-2}\, x_3 \;,\; C^{n-1}_{\boxed{2}}\, C^{n}_{n-1}\, x_1\, x_2^{n-3}\, x_3^{2} \;,\; \ldots\ldots\ldots\ldots \; C^{n-1}_{\boxed{n-2}}\, C^{n}_{n-1}\, x_1\, x_2\, x_3^{n-2}$$

We assume B_2denotes the number inside the closed curve where $B_2 = 2,3,\ldots\ldots\ldots\ldots,n-1$

We assume h denotes the number inside the rectangle where $h=1,2,\ldots\ldots\ldots\ldots,n-2$

At that point the terms can be written in the form

$$\sum_{B2=2,h=1}^{n-1,n-2} D_2\, x_1^{n-B2}\, x_2^{B2-h}\, x_3^{h}$$

Represents the coefficient of any term in the result $D_2 = C^{B2}_{h}\, C^{n}_{B2}$

The number of terms with three symbols in the result we will find= $\frac{(n-1)(n-2)}{2}$

we study $D_2\, x_1^{n-B2}\, x_2^{B2-h}\, x_3^{h}$

$$= C^{B2}_{h}\, C^{n}_{B_2}\, x_1^{n-B2}\, x_2^{B2-h}\, x_3^{h}$$

$$= \frac{\cancel{B2!}}{(B2-h)!h!} \cdot \frac{n!}{(n-B2)!\cancel{B2!}}\, x_1^{n-B2}\, x_2^{B2-h}\, x_3^{h}$$

We assume :

$n - B_2 = P_1$, $B_2 - h = p_2$, $h = p_3$

The result then becomes $\frac{n!}{p1!p2!p3!}\, x_1^{p1}\, x_2^{p2}\, x_3^{p3}$

where $p_1 + p_2 + p_3 = n$

$$(x_1 + x_2\; x_3)^n = \sum_{p1,p2,p3=0}^{n} \frac{n!}{p1!\,p2!p3!}\, x_1^{p1}\, x_2^{p2} x_3^{p3}$$

Third: we study m = 4

$(x_1+x_2+x_3+x_4)^n=[x_1+(x_2+x_3+x_4)]^n$

$=x_1^n+c_1^n x_1^{n-1}(x_2+x_3+x_4)+c_2^n x_1^{n-2}(x_2+x_3+x_4)^2+c_3^n x_1^{n-3}(x_2+x_3+x_4)^3+\ldots\ldots\ldots$

$+c_{n-1}^n x_1(x_2+x_3+x_4)^{n-1}+(x_2+x_3+x_4)^n$

We find in this result

$I_1=x_1^n$, $v_1=1$

$I_2=c_1^n x_1^{n-1}(x_2+x_3+x_4)=c_1^n x_1^{n-1}x_2+c_1^n x_1^{n-1}x_3+c_1^n x_1^{n-1}x_4$ $\qquad v_2=3$

$I_3=c_2^n x_1^{n-2}(x_2+x_3+x_4)^2=c_2^n x_1^{n-2}(x_2^2+x_3^2+x_4^2+c_1^2 x_2x_3+c_1^2 x_2x_4+c_1^2 x_3x_4)$

$=c_2^n x_1^{n-2}x_2^2+c_2^n x_1^{n-2}x_3^2+c_2^n x_1^{n-2}x_4^2+c_1^2 c_2^n x_1^{n-2}x_2x_3+c_1^2c_2^n x_1^{n-2}x_2x_4+c_1^2 c_2^n x_1^{n-2}x_3x_4$ $\qquad v_3=6$

$I_4=c_3^n x_1^{n-3}(x_2+x_3+x_4)^3=c_3^n x_1^{n-3}(x_2^3+x_3^3+x_4^3+c_1^3 x_2^2x_3+c_2^3 x_2x_3^2+c_1^3 x_2^2x_4+c_2^3 x_2x_4^2$

$+c_1^3 x_3^2x_4+c_2^3 x_3x_4^2+c_1^2c_2^3 x_2x_3x_4)$

$=c_3^n x_1^{n-3}x_2^3+c_3^n x_1^{n-3}x_3^3+c_3^n x_1^{n-3}x_4^3+c_1^3 c_3^n x_1^{n-3}x_2^2x_3+c_2^3 c_3^n x_1^{n-3}x_2x_3^2+c_1^3 c_3^n x_1^{n-3}x_2^2x_4$

$+c_2^3 c_3^n x_1^{n-3}x_2x_4^2+c_1^3c_3^n x_1^{n-3}x_3^2x_4+c_2^3 c_3^n x_1^{n-3}x_3x_4^2+c_1^2c_2^3c_3^n x_1^{n-3}x_2x_3x_4$ $\qquad v_4=10$

$I_5=c_4^n x_1^{n-4}(x_2+x_3+x_4)^4=c_4^n x_1^{n-4}(x_2^4+x_3^4+x_4^4+c_1^4 x_2^3x_3+c_2^4 x_2^2x_3^2+c_3^4 x_2x_3^3+c_1^4x_2^3x_4+$

$c_2^4 x_2^2x_4^2+c_3^4 x_2x_4^3+c_1^4 x_3^3x_4+c_2^4 x_3^2x_4^2+c_3^4 x_3x_4^3+c_1^2 c_2^4 x_2^2x_3x_4+c_1^3 c_3^4 x_2x_3^2x_4+$

$c_2^3 c_3^4 x_2x_3x_4^2)$

$=c_4^n x_1^{n-4}x_2^4+c_4^nx_1^{n-4}x_3^4+c_4^n x_1^{n-4}x_4^4+c_1^4 c_4^n x_1^{n-4}x_2^3x_3+c_2^4 c_4^n x_1^{n-4}x_2^2x_3^2+c_3^4 c_4^n x_1^{n-4}x_2x_3^3$

$+c_1^4 c_4^nx_1^{n-4}x_2^3x_4+c_2^4 c_4^n x_1^{n-4}x_2^2x_4^2+c_3^4 c_4^n x_1^{n-4}x_2x_4^3+c_1^4 c_4^n x_1^{n-4}x_3^3x_4+c_2^4c_4^n x_1^{n-4}x_3^2x_4^2$

$+c_3^4 c_4^n x_1^{n-4}x_3x_4^3+c_1^2 c_2^4c_4^n x_1^{n-4}x_2^2x_3x_4+c_1^3 c_3^4c_4^n x_1^{n-4}x_2x_3^2x_4+c_2^3 c_3^4 c_4^nx_1^{n-4}x_2x_3x_4^2$ $\qquad V_5=15$

The total number of terms= $v_1 + v_2 + v_3$ ……………………

$1+ 3 +6 + 10 + 15$ ………………… $= \sum_{n=0}^{n} \frac{(n+2)(n+1)}{2}$

$= c^{n+3}_{3} = c^{n+(4-1)}_{4-1} = c^{n+(m-1)}_{m-1}$

1-We find in the result four terms with one symbol are $x^4_1, x^4_2, x^4_3, x^4_4$

And write as

$\sum_{i=1}^{4} (x_i)^n$

2-We find in the result six types of terms with two symbols are

$x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4, x_3 x_4$

Each one of them has n-1 terms and the total sum is $c^{4}_{2}(n-1)$

can be written as $\sum_{r=1}^{n-1} D_1 (xi)^{n-r} (xj)^r$ i= 1,2,3 , j= 2,3,4

3- we find in the result four types of terms with three symbols are

$x_2 x_3 x_4$, $x_1 x_3 x_4$, $x_1 x_2 x_4$, $x_1 x_2 x_3$ for each one of them $\frac{(n-1)(n-2)}{2}$

of the term and their total sum $\frac{(n-1)(n-2)}{2} c^4_3$ can be written as

$\sum_{B2=2,h=1}^{n-1,n-2} D_2(x_i)^{n-B2} (xj)^{B_2-h} (x_k)^h$

4- we find in the result terms with four symbols $x_1 x_2 x_3 x_4$

in I_4 we find

$c^{2}_{\boxed{1}} \; c^{3}_{2} \; c^{n}_{3} \; x_1^{n-3} x_2 x_3 x_4$

In I_5 we find

$c^{2}_{\boxed{1}} \; c^{4}_{2} \; c^{n}_{4} \; x_1^{n-4} x_2^2 x_3 x_4$, $c^{3}_{\boxed{1}} \; c^{4}_{3} \; c^{n}_{4} \; x_1^{n-4} x_2 x_3^2 x_4$, $c^{3}_{\boxed{2}} \; c^{4}_{3} \; c^{n}_{4} \; x_1^{n-4} x_2 x_3 x_4^{2}$

the number inside the closed curve B2 where $B_2 = 2,3, \ldots\ldots\ldots,n-2$

the number inside the rectangle h where $h= 1,2,\ldots\ldots\ldots\ldots, n-3$

we assume the number inside the triangle B3 where $B_3= 3,4\ldots\ldots\ldots\ldots, n-1$

in any term we will find that

X_1 power = $n-B_3$

X_2 power $=B_3-B_2$

X_3 power = B_2-h

X_4 power $=h$

can be written as

$$\sum_{B3=3\ ,B2=2\ ,h=1}^{n-1,n-2\ ,n-3} \qquad D_3\, X_1^{n-B3}\, X_2^{B3-B2}\, X_3^{B2-h}\, X_4^{h}$$

$$D_3=C_h^{B2}\ \ C_{B2}^{B3}\ \ C_{B3}^{n}$$

Study:

$$D_3\, X_1^{n-B3}\, X_2^{B3-B2} X_3^{B2-h}\, X_4^{h}$$

$$= C_h^{B2}\, C_{B2}^{B3}\, C_{B3}^{n}\, X_1^{n-B3}\, X_2^{B3-B2}\, X_3^{B2-h}\, X_4^{h}$$

$$\frac{\cancel{B2!}}{h!(B2-h)!} \cdot \frac{\cancel{B3!}}{\cancel{B2!}(B3-B2)!} \cdot \frac{n!}{\cancel{B3!}(n-B3)!} \qquad X_1^{n-B3}\, X_2^{B3-B2}\, X_3^{B2-h}\, X_4^{h}$$

we assume $n-B_3=P_1$, $B_3-B_2=P_2$, $B_2-h=p_3$, $h=p_4$

so the result $\frac{n!}{p1!p2!p3!p4!} \quad x_1^{P_1}\ x_2^{P_2}\ x_3^{P_3} x_4^{P_4}$

where $p_1+ p_2+ p_3+ p_4=n$

and it will be

$$(x_1+ x_2+ x_3+x_4)^n = \sum_{p1,p2,p3,p4=0}^{n} \frac{n!}{p1!p2!p3!p4!} \quad x_1^{P_1}\ x_2^{P_2}\ x_3^{P_3}\ x_4^{P_4}$$

using the same method we conclude

$$(x_1+x_2+x_3+\ldots\ldots+x_m)^n = \sum_{p1,p2,p3\ldots pm=0}^{n} \frac{n!}{p1!,p2!,\ldots\ldots pm!} x_1^{P_1} x_2^{P_2} x_3^{P_3} \ldots x_m^{p_m}$$

where $p_1+p_2+p_3\ldots\ldots\ldots\ldots+p_m=n$

Note: the results of this expansion are calculated in two orders

1- the first order is according to the symbols included in each term, meaning terms with one symbol $X_1, X_2, \ldots\ldots\ldots X_m$

two-symbol terms $\ldots\ldots\ldots\ldots, X_2 X_3, X_2 X_4, \ldots\ldots, X_2 X_m, X_1 X_2, X_1 X_3, \ldots\ldots. X_1 X_m$

three-symbol terms, four-symbol terms, ...

2- the second order is according to the exponents of the symbols included in the term x_1 then x_2, …………… descending (Newton's style in the binomial expansion)

for example: $(x + y + z)^5$
solution:

the general term $= \dfrac{n!}{p1!p2!p3!} x_1^{p1} x_2^{p2} x_3^{p3}$

the first method: T is the symbol of the term

$T_1=\frac{5!}{5!0!0!} x^5 y^0 z^0 = x^5$, $T_2=\frac{5!}{0!5!0!} x^0 y^5 z^0 = y^5$

$T_3=\frac{5!}{0!0!5!} x^0 y^0 z^5 = z^5$, $T_4=\frac{5!}{4!1!0!} x^4 y z^0 = 5x^4 y$

$T_5=\frac{5!}{3!2!0!} x^3 y^2 z^0 = 10x^3 y^2$, $T_6=\frac{5!}{2!3!0!} x^2 y^3 z^0 = 10x^2 y^3$

$T_7=\frac{5!}{1!4!0!} x y^4 z^0 = 5x y^4$, $T_8=\frac{5!}{4!0!1!} x^4 y^0 z = 5x^4 z$

$T_9=\frac{5!}{3!0!2!} x^3 y^0 z^2 = 10 x^3 z^2$, $T_{10}=10x^2 z^3$, $T_{11}= 5x z^4$, $T_{12}= 5y^4 z$

$T_{13}= 10\ y^3z^2$, $T_{14}= 10y^2 z^3$, $T_{15}= 5\ y\ z^4$

$T_{16} =\frac{5!}{3!1!1!}\ x^3\ y\ z\ =20\ x^3\ y\ z$

$T_{17} =\frac{5!}{2!2!1!}\ x^2\ y^2\ z\ = 30\ x^2\ y^2\ z$, $T_{18}= \frac{5!}{2!1!2!}\ x^2\ y\ z^2 = 30\ x^2\ y\ z^2$

$T_{19}=\frac{5!}{1!3!1!}\ x\ y^3\ z\ = 20\ x\ y^3\ z$, $T_{20}=\frac{5!}{1!2!2!}\ x\ y^2z^2 =30\ x\ y^2z^2$

$T_{21}=\frac{5}{1!1!3!}\ x\ y\ z^3\ =20\ x\ y\ z^3$

<u>second method</u>:

$T_1=\frac{5!}{5!0!0!}\ x^5\ y^0\ z^0\ =\ x^5$, $T_2= \frac{5!}{4!1!0!}\ x^4\ y\ z^0 = 5\ x^4y$

$T_3=\frac{5!}{4!0!1!}\ x^4\ y^0 z\ =5\ x^4 z$, $T_4= \frac{5!}{3!2!0!}\ x^3\ y^2\ z^0 =10\ x^3 y^2$

$T_5=\frac{5!}{3!1!1!}\ x^3\ y\ z =\ 20\ x^3\ y\ z$, $T_6 = \frac{5!}{3!0!2!}\ x^3\ y^0\ z^2 =10\ x^3\ z^2$

$T_7= \frac{5!}{2!3!0!}\ x^2 y^3\ z^0 =10\ x^2\ y^3$, $T_8= \frac{5!}{2!2!1!}\ x^2 y^2\ z =30\ x^2 y^2\ z$

$T_9= \frac{5!}{2!1!2!}\ x^2 y\ z^2 = 30\ x^2\ y\ z^2$, $T_{10}=\frac{5!}{2!0!3!}\ x^2\ y^0 z^3 =10\ x^2\ z^3$

$T_{11}=\frac{5!}{1!4!0!}\ x\ y^4z^0\ =5\ x\ y^4$, $T_{12}= \frac{5!}{1!3!1!}\ x\ y^3\ z=20\ x\ y^3\ z$

$T_{13}=\frac{5!}{1!2!2!}\ x\ y^2\ z^2 = 30\ x\ y^2 z^2$, $T_{14}= \frac{5!}{1!1!3!}\ x\ y\ z^3 =20\ x\ y\ z^3$

$T_{15}=\frac{5!}{1!0!4!}\ x\ y^0\ z^4\ = 5x\ z^4$, $T_{16}= \frac{5!}{0!5!0!}\ x^0\ y^5\ z^0 = y^5$

$T_{17}=\frac{5!}{0!4!1!}\ x^0 y^4\ z =5\ y^4\ z$, $T_{18}= \frac{5!}{0!3!2!} x^0 y^3\ z^2\ =10\ \ y^3z^2$

$T_{19}=\frac{5!}{0!2!3!}\ x^0\ y^2\ z^3\ = 10\ y^2 z^3$, $T_{20}=\frac{5!}{0!1!4!}\ x^0\ y\ z^4 =5\ yz^4$

$T_{21}=\frac{5!}{0!0!5!}\ x^0\ y^0\ z^5\ =Z^5$

example in expansion $(x+ y+ z+ L)^n$

one of the terms was found $k\, x^3 y^2 z\, L$,where (k) is the numerical coefficient

1-find the values of (n, k)

2- find the number of terms in the expansion

3-write the terms that contain the four symbols (x, y, z, L)

solution: in the term $k\, x^3\, y^2\, z\, L$

1 - $P_1=3$, $P_2=2$, $P_3=1$, $P_4=1$

$\therefore\ n=p_1+p_2+p_3+p_4=7$

2- the numerical coefficient (k) $= \frac{n!}{p1!p2!p3!p4!} = \frac{7!}{3!2!1!1!} = 420$

number of terms $= C^{n+m-1}_{m-1} = C^{7+4-1}_{4-1} = C^{10}_{3} = 120$

3 - number of terms with one symbol m=4

number of terms with two symbols $= c^m_2(n-1) = c^4_2(7-1) =36$

number of terms with three symbols $= c^m_3 \frac{(n-1)(n-2)}{2} = c^4_3 \frac{(6)(5)}{2} = 60$

number of terms with four symbols $= c^{10}_3 - (\ 2\, n^2+2) = 20$

terms with four symbols are:

$T_1= \frac{7!}{4!1!1!1!}\ x^4\, y\, z\, L = 210\ x^4\, y\, z\, L$, $T_2= \frac{7!}{3!2!1!1!}\ x^3\, y^2\, z\, L = 420\ x^3\, y^2\, z\, L$

$T^3=\frac{7!}{3!1!2!1!}\ x^3\, y\, z^2\, L = 420\ x^3 y\, z^2\, L$,$T_4= \frac{7!}{3!1!1!2!}\ x^3\, y\, z\, L^2 = 420\, x^3\, y\, z\, L^2$

$T_5 =\frac{7!}{2!3!1!1!}\ x^2\, y^3\, z\, L =420\ x^2\, y^3\, z\, L$, $T_6= \frac{7!}{2!2!2!1!}\, x^2\, y^2\, z^2\, L= 630\, x^2\, y^2 z^2\, L$

$T_7= \frac{7!}{2!2!1!2!}\ x^2\, y^2\, z\, L^2 = 630\ x^2\, y^2\, z\, L^2$,$T_8=\frac{7!}{2!1!3!1!}\ x^2\, y\, z^3 L = 420x^2\, y\, z^3\, L$

$T_9= \frac{7!}{2!1!2!2!}\ x^2\, y\, z^2\, L^2 = 630\ x^2\, y\, z^2\, L^2$, $T_{10}= \frac{7!}{2!1!1!3!}\ x^2\, y\, z\, L^3 =420\, x^2\, y\, z\, L^3$

$T_{11} = \frac{7!}{1!4!1!1!}$ x y^4 z L = 210 x y^4 z L , $T_{12}= \frac{7!}{1!3!2!1!}$ x $y^3 z^2$ L =420 x $y^3 z^2$ L

$T_{13} = \frac{7!}{1!3!1!2!}$ x y^3 z L^2 =420 x y^3 z L^2 , $T_{14}= \frac{7!}{1!2!3!1!}$ x $y^2 z^3$ L = 420 x $y^2 z^3$ L

$T_{15}=\frac{7!}{1!2!2!2!}$ x $y^2 z^2 L^2$ = 630 x $y^2 z^2 L^2$, $T_{16}= \frac{7!}{1!2!1!3!}$ x y^2 z L^3 = 420 x y^2 z L^3

$T_{17}= \frac{7!}{1!1!4!1!}$ x y z^4 L = 210 x y z^4 L , $T_{18}= \frac{7!}{1!1!3!2!}$ x y $z^3 L^2$ =420 x y $z^3 L^2$

$T_{19}=\frac{7!}{1!1!3!2!}$ x y $z^3 L^2$ = 420 x y $z^3 L^2$, $T_{20}= \frac{7!}{1!1!1!4!}$ x y z L^4= 210x y z L^4

example:- in expansion $(x^2+ y+1)^7$

find T_5 , T_{18} , T_{22} , T_{30}

Solution:

number of terms = C^{n+m-1}_{m-1} = C^{9}_{2} = 36

3 term are x^n_1, x^n_2, x^n_3, 6 term for $x_1 x_2$, 6 term for $x_1 x_3$, 6 term for $x_2 x_3$

15 Term for $x_1 x_2 x_3$

So T_5 represents the second term in the group $x_1 x_2$, with $p_1 = 5$, $p_2 = 2$, and $p_3 = 0$

$$T_5=\frac{7!}{5!2!0!} (x^2)^5 (y)^2 (1)^0 = 28 x^{10} y^2$$

T_{18} represents the third term in the group $x_2 x_3$, with $p_1 = 0$, $p_2 = 4$, and $p_3 = 3$

$$T_{18} = \frac{7!}{0!4!3!} (x^2)^0 (y)^4 (1)^3 = 35y^4$$

T_{22} represents the first term in the group $x_1 x_2 x_3$, with $p_1 = 5$, $p_2 = 1$, and $p_3 = 1$

$$T_{22} =\frac{7!}{5!1!1!} (x^2)^5(y)(1) =42 x^{10}y$$

T_{30} represents the ninth term in the group $x_1 x_2 x_3$, with $p_1 = 2$, $p_2 = 2$, and $p_3 = 3$

$$T_{30}=\frac{7!}{2!2!3!} (x^2)^2 (y)^2 (1)^3 = 210 x^4 y^2$$

example: In the expansion of, $(x^2+ x+1 +\frac{1}{x})^{10}$ find the coefficient of the term containing x^{14}

solution:

1- The group with a single symbol $(x^2)^{10}$, $(x)^{10}$, $(1)^{10}$, $(\frac{1}{x})^{10} \neq x^4$ ignore

2- The group with two symbols

First: in

$x_1x_2 \rightarrow p_1 + p_2 = 10$, and $2p_1 + p_2 = 14$ Solve the two equations

$p_1=4$, $p_2=6$ $\therefore$ Coefficient $\frac{10!}{4!6!} = 210$

Second: in $x_1x_3 \rightarrow p_1 + p_3 = 10$, $2p_1 = 14 \rightarrow p_1 = 7$, $p_3 = 3$

Coefficient = $\frac{10!}{7!3!} = 120$

Third: in $x_1x_4 \rightarrow p_1 + p_4 = 10$, $2p_1 - p_4 = 14 \rightarrow p_1 = 8$, $p_4 = 2$

Coefficient = $\frac{10!}{8!2!} = 45$

Fourth: in $x_2x_3 \rightarrow p_2 + p_3 = 10$, $p_2 = 14$ ignore

Fifth: in $x_2x_4 \rightarrow p_2 + p_4 = 10$, $p_2 - p_4 = 14 \rightarrow p_2 = 12$ ignore

Sixth: in $x_3x_4 \rightarrow p_3 + p_4 = 10$, $-p_4 = 14$ ignore

3- The group with three symbols

in $x_1x_2x_3 \rightarrow p_1 + p_2 + p_3 = 10$, $2p_1 + p_2 = 14$, this holds when

$p_1 = 5$, $p_2 = 4$, $p_3 = 1$ $\therefore$ coefficient $\frac{10!}{5!\,4!\,1!} = 1260$

in $x_1x_2x_4 \rightarrow p_1 + p_2 + p_4 = 10$, $2p_1 + p_2 - p_4 = 14$, this holds when

$p_1 = 6$, $p_2 = 3$, $p_4 = 1$ $\therefore$ coefficient $=\frac{10!}{6!3!1!} = 840$

in $x_1x_3x_4 \rightarrow p_1 + p_3 + p_4 = 10$, $2p_1 - p_4 = 14$, by addition $3p_1 + p_3 = 24$
no values satisfy → ignore
in $x_2x_3x_4 \rightarrow p_2 + p_3 + p_4 = 10$, $p_2 - p_4 = 14$, by addition $2p_2 + p_3 = 24$
no values satisfy → ignore

4- The terms with four symbols $x_1x_2x_3x_4$

$p_1 + p_2 + p_3 + p_4 = 10$, $2p_1 + p_2 - p_4 = 14$, by addition

This holds when $3p_1+2p_2+p_3=24$

$p_1 = 7$, $p_2 = 1$, $p_3 = 1$, $p_4 = 1$

coefficient = $\frac{10!}{7!1!1!1!}$ = 720

The coefficient of the term containing x^{14} = 210+120+45+1260+840+720 = 3195

Discussion

The present study proposed a generalized mathematical formula for expanding the polynomial $(x_1 + x_2 + \cdots + x_m)^n$, extending the well-known Binomial Theorem from two variables to any finite number of variables. By systematically analyzing the cases $m = 2$, $m = 3$, and $m = 4$, common structural patterns were identified and generalized into a unified expression applicable for arbitrary values of m.The analysis of the case $m = 2$confirmed the validity of Newton's Binomial Theorem and demonstrated that the expansion consists of $n + 1$terms with coefficients represented by the binomial coefficients. This case served as the mathematical foundation for extending the expansion to higher-dimensional polynomials. When the number of variables was increased to three $(m = 3)$, the expansion exhibited a more complex structure involving terms containing one, two, and three variables. The total number of terms was shown to be

$$\frac{(n+2)(n+1)}{2},$$

which corresponds to the trinomial coefficient formula. The derivation also revealed systematic relationships among the coefficients and exponents, allowing the terms to be classified according to the number of variables involved. This classification simplified the expansion process and highlighted the combinatorial nature of multinomial expressions. The study of the case

$m = 4$ further confirmed the consistency of the observed patterns. The expansion generated additional categories of mixed-variable terms while preserving the same combinatorial structure identified for smaller values of m. This demonstrated that the coefficients follow a predictable arrangement governed by combinations and factorial expressions rather than requiring independent derivations for each value of m.Based on these observations, a general multinomial expansion formula was established. The coefficient of each term was expressed using factorial notation,

$$\frac{n!}{p_1!\,p_2!\cdots p_m!},$$

subject to the condition

$$p_1 + p_2 + \cdots + p_m = n.$$

This expression provides a direct method for determining both the coefficients and the exponents of every term in the expansion without performing repeated multiplication.

The proposed method offers a systematic and efficient alternative for expanding high-order multivariable polynomials. Rather than deriving each expansion individually, the generalized formula enables direct computation for any number of variables, significantly reducing computational complexity and simplifying symbolic calculations. Consequently, the derived formula represents a natural extension of Newton's Binomial Theorem and provides a practical mathematical framework for applications in algebra, combinatorics, probability, and computational mathematics.

Conclusion

This study developed a generalized formula for expanding, $(x_1 + x_2 + \dots + x_m)^n$, extending Newton's Binomial Theorem to any finite number of variables. Analysis of the cases m = 2, m = 3, and m = 4 revealed consistent combinatorial patterns that led to the multinomial coefficient formula,

$$\frac{n!}{p1!p2!\dots pm!}$$

where $p_1 + p_2 + \dots + p_m = n$. The proposed formula provides a direct and efficient method for determining all terms of the expansion, reducing computational complexity while simplifying symbolic calculations. This generalized approach has broad applications in algebra, combinatorics, probability, and computational mathematics.

Conflicts of interest

The authors declare that they have no conflicts of interest related to this work.

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Data availability statement

The datasets generated and analyzed during the current study are available from the corresponding author upon reasonable request.

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