

Approximate Solutions of Equations in Numerical Analysis

Oday Hatem Jalel

(Director General of Education in Babylon Governorate, Iraq)

Abstract:

Numerical analysis gives useful ways of solving mathematical equations where exact solutions are either hard to obtain or not possible. Algebraic equations, systems of equations and differential equations are used to express many problems in engineering, physics, computer science, economics and applied sciences. Real world models are often complicated and therefore, approximate solutions with acceptable accuracy are obtained using numerical methods. In this article the use of numerical analysis in solving equations is discussed and approximation, error, convergence, stability, and applications are touched upon. It also describes the relevance of choosing appropriate numerical techniques based on the type of the equation and the desired accuracy.

Keywords: Numerical analysis, equations, approximation, error, convergence, stability, numerical methods.

1. Introduction

Numerical analysis is a field in applied mathematics that deals with the study and development of algorithms for the approximation of mathematical problems. It is particularly important when equations can't be solved exactly analytically. In many scientific and engineering problems mathematical equations are used to describe real phenomena, but the equations may be too complicated to solve by hand. Thus, numerical analysis offers useful methods to transform mathematical models into numerical solutions [1]. The primary objective of numerical analysis is not only to obtain a solution, but also to quantify the accuracy and reliability of the solution. While a numerical solution can be very close to an exact solution, there is always some amount of error associated with it. In this regard, the numerical analysis studies the error estimation, stability, convergence, and computational efficiency [2]. These ideas help researchers to select the most appropriate method for every mathematical problem.

2. Solving Nonlinear Equations

One common problem in numerical analysis is solving nonlinear equations of the form:

$$f(x) = 0$$

These equations are found in physics, optimization, economics and in engineering design. Numerical methods for root finding are employed since there are no simple exact solutions to the nonlinear equations.

These include bisection method, fixed-point iteration, secant method and Newton's method [3].

The bisection method is easy to use and reliable. It works when a continuous function has a sign change on an interval. This process is repeated until the root is approximated. This method is slow, but is stable and simple. However, Newton's method will be quicker if the starting approximation is close to the actual root. It is written as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This formula is based on the derivative of the function to improve the approximation. But, when the derivative is very small or when the initial guess is not appropriate, Newton's method may not work [4]. Hence, the success of a numerical method will be dependent on the mathematical structure of the equation and on the conditions under which the method is applied.

3. Error and Accuracy

Errors are of fundamental concern in numerical analysis. However, numerical methods provide only approximate solutions, and the researcher should be aware of the possible amounts of error in the solution. There are two major types of numerical error: truncation error and round-off error. Truncation error: An error that arises when an infinite mathematical process is approximated by a finite one. Round-off error is an error which arises due to the fact that computers represent numbers only up to a certain degree of accuracy [5].

For example, a derivative may be approximated numerically by:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

A larger step size will give a less accurate approximation. If too small, round-off error might increase. Thus, one needs to have a balance for numerical accuracy. A satisfactory numerical method would be one that would reduce the error without significantly increasing the computational burden [6].

Additionally, accuracy is related to the stopping criterion. With iterative methods, the iteration process is continued until a sufficiently small difference between two successive approximations is achieved. However, terminating too early may result in an incorrect answer and going on for too long may be a waste of computational time.

4. Systems of Linear Equations

Many applied problems lead to systems of linear equations written as:

$$Ax = b$$

where A is a matrix, x is the unknown vector, and b is the known vector. Linear systems are used in structural analysis, electrical circuits, computer graphics, numerical simulations, and data science [7].

There are basically two methods of solving a linear system – direct and iterative. Direct methods (Gaussian elimination etc.) attempt to find the solution in a finite number of operations. They can be

employed in small and medium sized systems. But if the system is very large then direct methods might consume too much memory and time.

Iterative methods are better suited for large systems, in particular if the matrix is sparse. These are methods that start with an approximation and make successive approximations. Classical examples are the Jacobi method, the Gauss-Seidel method and the relaxation methods [8]. A higher-level technique is the conjugate gradient technique which is commonly used for symmetric positive definite systems [9]. Its significance is based on its efficiency in solving large linear systems which occur in scientific computing.

5. Differential Equations

One of the most important equations in the field of applied mathematics are differential equations. They appear in various applications including heat transfer, population growth, fluid flow, mechanics and electrical systems to mention a few. Many differential equations cannot be solved exactly and numerical methods are needed [10].

A basic ordinary differential equation may be written as:

$$\frac{dy}{dt} = f(t, y)$$

Simple methods, like Euler's method, give an approximation to the solution in steps. More accurate methods are widely used such as Runge-Kutta methods which give greater accuracy without the need of very small step values. Embedded Runge-Kutta formulas are particularly convenient since they provide an approximation for the error and self-adjust the step size [11].

Partial differential equations are more difficult because there are multiple independent variables. The Crank-Nicolson method is a popular numerical technique for heat-conduction problem and other time-dependent equations [12]. These approaches are used to convert continuous equations into a set of algebraic equations that can be solved by the computer.

6. Stability and Convergence

Any numerical method should be stable and convergent. Stability implies that the error made during the computation should not be excessive, but should not be growing in size. Convergence occurs when the numerical solution gets closer to the exact solution as the step size decreases and/or the number of iteration steps increases.

Both of these concepts are crucial to numerical analysis. Mathematically correct method can be found, which will give poor results if it is unstable. Likewise, a method that converges sufficiently quickly can be unstable but without an accurate solution. Thus, the numerical methods are judged based on accuracy, stability, convergence rate and cost [2] [6].

The best approach isn't always the most complicated one in practical applications. Sometimes a very simple method is desired, because it is easy to execute and is accurate enough. In other situations, more sophisticated techniques are needed where the problem is large or nonlinear, or sensitive to error.

7. Applications

Numerous applications today use numerical analysis. It is used by engineers to study buildings, bridges, machines and electrical systems. It is used by physics to represent waves, particles and heat transfer. There are numerical models to help economists predict financial behavior. Computer scientists use numerical methods in the fields of machine learning, image processing, graphics and optimization.

The advantage of numerical analysis is that it is able to bring the theory to practice. Mathematical equations are used to describe real-world problems and numerical methods make the equations computer solvable. Many uses of science and technology would fail to be done efficiently if they did not conduct a numerical analysis.

8. Conclusion

In science and engineering, numerical analysis is an important area of study to solve equations. It offers practical methods for nonlinear equations, linear systems and differential equations. Sometimes there are no exact solutions available and numerical methods give approximate solutions which can be controlled and evaluated.

The properties of accuracy, error control, stability, convergence and computational efficiency are key for the quality of a numerical solution. Hence, numerical analysis is not just a calculation process but also a scientific judging method. Assists researchers in selecting appropriate algorithms, recognizing limitations, and achieving accurate results with complex mathematical models.

References

- [1] S. D. Conte and C. de Boor, Elementary Numerical Analysis: An Algorithmic Approach, 3rd ed. New York, NY, USA: McGraw-Hill, 1980.
- [2] M. T. Heath, Scientific Computing: An Introductory Survey, 2nd ed. Philadelphia, PA, USA: SIAM, 2002.
- [3] A. Ralston and P. Rabinowitz, A First Course in Numerical Analysis, 2nd ed. New York, NY, USA: McGraw-Hill, 1978.
- [4] J. M. Ortega and W. C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables. New York, NY, USA: Academic Press, 1970.
- [5] F. S. Acton, Numerical Methods That Work. Washington, DC, USA: Mathematical Association of America, 1990.
- [6] G. E. Forsythe, M. A. Malcolm, and C. B. Moler, Computer Methods for Mathematical Computations. Englewood Cliffs, NJ, USA: Prentice-Hall, 1977.
- [7] E. Isaacson and H. B. Keller, Analysis of Numerical Methods. New York, NY, USA: Wiley, 1966.
- [8] D. M. Young, Iterative Solution of Large Linear Systems. New York, NY, USA: Academic Press, 1971.
- [9] M. R. Hestenes and E. Stiefel, “Methods of conjugate gradients for solving linear systems,” Journal of Research of the National Bureau of Standards, vol. 49, no. 6, pp. 409–436, 1952.
- [10] P. Henrici, Discrete Variable Methods in Ordinary Differential Equations. New York, NY, USA: Wiley, 1962.
- [11] J. R. Dormand and P. J. Prince, “A family of embedded Runge-Kutta formulae,” Journal of Computational and Applied Mathematics, vol. 6, no. 1, pp. 19–26, 1980.
- [12] J. Crank and P. Nicolson, “A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type,” Proceedings of the Cambridge Philosophical Society, vol. 43, no. 1, pp. 50–67, 1947.